A determinant formula for relative congruence zeta functions for cyclotomic function fields

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Abstract

In the paper [Ro], Rosen gave a determinant formula for relative class numbers for the P-th cyclotomic function fields in the case of the monic irreducible polynomial P, which is regarded as an analogue of the classical Maillet determinant. In this paper, we will give a determinant formula for the relative congruence zeta functions for cyclotomic function fields. Our formula is regarded as a generalization of the determinant formula for the relative class number.

1 Introduction

Let h_p^- be the relative class number of cyclotomic field of p-th root of unity. In the paper [C-O], Carlitz and Olson computed the number h_p^- in terms of a certain classical determinant, which is known as the Maillet determinant.

In the cyclotomic function field case, several authors gave an analogue of Maillet determinants.

Let k be a field of rational functions over a finite field $\mathbb{F}_q$ with q elements. Fix a generator T of k, and let $A = \mathbb{F}_q[T]$ be the polynomial subring of k. Let m be a monic polynomial of A, and Λ_m be the set of all of m-torsion points of the Carlitz module. The field K_m obtained by adding the points of Λ_m to k is called the m-th cyclotomic function field. For the definition of Carlitz module and basic facts of cyclotomic function fields, see Section 2

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below. Let K_m^+ be the decomposition field of the infinite prime of k in K_m/k, which is called the "maximal real subfield" in K_m.

Let h_m, h_m^+ be orders of the divisor class group of degree 0 for K_m, and K_m^+, respectively. Define the relative class number h_m^- of K_m by $h_m^- = h_m/h_m^+$.

Rosen gave a determinant formula for h_P^- in the case of the monic irreducible polynomial P (cf. [Ro]), which is regarded as an analogue of the Maillet determinant. Recently, several authors generalized the Rosen's formula and gave class number formulas. (cf. [B-K], [A-C-J]).

Let $\zeta(s, K_m)$ be the congruence zeta function for K_m. The function $\zeta(s, K_m)$ can be expressed by

$$\zeta(s, K_m) = \frac{P_m(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $P_m(X)$ is a polynomial with integral coefficients. Then we have the decomposition $P_m(X) = P_m^{(+)}(X)P_m^{(-)}(X)$, where $P_m^{(+)}(X)$ is the polynomial corresponding to the congruence zeta function $\zeta(s, K_m^+)$ for K_m^+. On the polynomial $P_m^{(+)}(X)$, the author gave the determinant formula in the paper [Sh]. We see that $P_m^{(-)}(q^{-s}) = \zeta(s, K_m)/\zeta(s, K_m^+)$, which is called the relative congruence zeta function for K_m.

The main result of the present paper is to give the determinant formula for $P_m^{(-)}(X)$. Since $P_m^{(-)}(1) = h_m^-$, our formula is regarded as a generalization of the determinant formula for the relative class number.

As an application of our determinant formula, we will give an explicit formula for some coefficients of low degree terms for $P_m^{(-)}(X)$.

2 Basic facts

In this section, we will provide several basic facts of cyclotomic function fields and its zeta functions. For the proof of these facts, see [G-R], [Ro 2], [Wa].

2.1 Cyclotomic function fields

Let K^{ac} be the algebraic closure of k. For $x \in K^{ac}$ and $m \in A$, we define the following action:

$$m \cdot x = m(\varphi + \mu)(x), \tag{1}$$

where $\varphi,\ \mu$ are $\mathbb{F}_q$-linear maps of K^{ac} defined by

$$\begin{aligned} \varphi : K^{ac} &\longrightarrow K^{ac} \qquad (x \mapsto x^q), \\ \mu : K^{ac} &\longrightarrow K^{ac} \qquad (x \mapsto T \cdot x). \end{aligned}$$

By the above action, K^{ac} becomes a A-module, which is called the Carlitz module. Let Λ_m be the set of all x satisfying $m \cdot x = 0$, which is a cyclic sub-A-module of K^{ac}. Fix a generator λ_m of Λ_m. Then we have the following isomorphism of A-modules

$$A/(m) \longrightarrow \Lambda_m \ (a \mod m \mapsto a \cdot \lambda_m), \tag{2}$$

where $(m) = mA$ is principal ideal generated by m. Let $(A/(m))^\times$ be the unit group of $A/(m)$, and $\Phi(m)$ be the order of $(A/(m))^\times$. Let K_m be the field obtained by adding elements of Λ_m to k. We call K_m the m-th cyclotomic function field. The extension K_m/k is an abelian extension, and we get the following isomorphism

$$(A/(m))^\times \longrightarrow \mathrm{Gal}(K_m/k) \ (a \bmod m \mapsto \sigma_{a \bmod m}) \tag{3}$$

where $\mathrm{Gal}(K_m/k)$ is the Galois group of K_m/k, and $\sigma_{a \bmod m}$ is the isomorphism given by $\sigma_{a \bmod m}(\lambda_m) = a \cdot \lambda_m$. By using the above isomorphism, we find that the extension degree of K_m/k is $\Phi(m)$.
We see that $\mathbb{F}_q^\times$ is contained in $(A/(m))^\times$. Let K_m^+ be the subfield of K_m corresponding to $\mathbb{F}_q^\times$. Again by the isomorphism (3), we find that the extension degree of K_m^+/k is $\Phi(m)/(q-1)$. Let P_∞ be the unique prime of k which corresponds to the valuation v_∞ with $v_\infty(T) < 0$. The prime P_∞ splits completely in K_m^+/k, and any prime of K_m^+ over P_∞ is totally ramified in K_m/K_m^+. Hence $K_m^+ = K_m \cap k_\infty$ where k_∞ is the completion of k by v_∞. The field K_m^+ is called the maximal real subfield of K_m, which is an analogue of maximal real subfields of cyclotomic fields.

Next, we provide basic facts about Dirichlet characters. For a monic polynomial $m \in A$, let X_m be the group of all primitive Dirichlet characters of $(A/(m))^\times$. Let X_m^+ be the set of characters contained in X_m such that $\chi(a) = 1$ for any $a \in \mathbb{F}_q^\times$. Put

$$\widetilde{K} = \bigcup_{m:\text{monic}} K_m \tag{4}$$

where m runs through all monic polynomials of A. Let $\mathbb{D}$ be the group of all primitive Dirichlet characters. By the same argument as in Chapter 3

in [Wa], we have a one-to-one correspondence between finite subgroups of $\mathbb{D}$ and finite subextension fields of $\widetilde{K}/k$. The following theorem is useful to obtain the imformation of primes.

Theorem 2.1. (cf. [Wa], Theorem 3.7.) *Let X be a finite subgroup of $\mathbb{D}$, and K_X the associated field. For a irreducible monic polynomial $P \in A$, put*

$$Y = \{\chi \in X \mid \chi(P) \neq 0\}, \quad Z = \{\chi \in X \mid \chi(P) = 1\}.$$

Then, we have

$$\begin{aligned}
X/Y &\simeq \text{ the inertia group of } P \text{ of } K_X/k \ ,\\
Y/Z &\simeq \text{ the cyclic group of order } f_P,\\
X/Z &\simeq \text{ the decomposition group of } P \text{ for } K_X/k,
\end{aligned}$$

where f_P is the residue class degree of P in K_X/k.

2.2 The relative congruence zeta function

Our next task is to investigate the congruence zeta function for cyclotomic function fields.

Let K be the geometric extension of k of finite degree. We define the congruence zeta function of K by

$$\zeta(s, K) = \prod_{\mathcal{P}:\text{prime}} \left(1 - \frac{1}{\mathcal{N}\mathcal{P}^s}\right)^{-1} \tag{5}$$

where $\mathcal{P}$ runs through all primes of K, and $\mathcal{N}\mathcal{P}$ is the number of elements of the reduce class field of a prime $\mathcal{P}$. We see that $\zeta(s, K)$ converges absolutely for $\mathrm{Re}(s) > 1$.

Theorem 2.2. *Let g_K be the genus of K and h_K be the order of divisor class group of degree 0. Then, there is a polynomial $P_K(X) \in \mathbb{Z}[X]$ of degree $2g_K$ satisfying*

$$\zeta(s, K) = \frac{P_K(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}, \tag{6}$$

and $P_K(0) = 1,\ P_K(1) = h_K$.

Since the right-handside of equation (6) is meromorphic on the whole of $\mathbb{C}$, this equation provides the analytic continuation of $\zeta(s,K)$ to the whole of $\mathbb{C}$.

Next, we explain the zeta function of $\mathcal{O}_K$, which is the integral closure of A in the field K. We define the zeta function $\zeta(s,\mathcal{O}_K)$ for the ring $\mathcal{O}_K$ by

$$\zeta(s,\mathcal{O}_K)=\prod_{\mathcal{P}}\Big(1-\frac{1}{\mathcal{N}\mathcal{P}^s}\Big)^{-1} \tag{7}$$

where the product runs over all primes of $\mathcal{O}_K$.

Let X be a finite subgroup of $\mathcal{D}$, and K_X be the associated field. By the same argument as in the case of number fields (cf. [Wa]), we have the following decomposition by L-functions

$$\zeta(s,\mathcal{O}_{K_X})=\prod_{\chi\in X}L(s,\chi) \tag{8}$$

where the L-function is defined by $L(s,\chi)=\prod_{P}\Big(1-\dfrac{\chi(P)}{\mathcal{N}P^s}\Big)^{-1}$ with P running through all monic irreducible polynomials of A.

Let f_∞, g_∞ be the residue class degree of P_∞ in K_X/k, and the number of prime in K_X over P_∞, respectively. Then we have

$$\zeta(s,K_X)=\zeta(s,\mathcal{O}_{K_X})(1-q^{-sf_\infty})^{-g_\infty}. \tag{9}$$

From now on, we will focus on cyclotomic function field case. For a monic polynomial $m\in A$, let K_m, K_m^+ be the m-th cyclotomic function field and its maximal real subfield. The relative congruence zeta function $\zeta^{(-)}(s,K_m)$ is defined by

$$\zeta^{(-)}(s,K_m)=\frac{\zeta(s,K_m)}{\zeta(s,K_m^+)}. \tag{10}$$

By Theorem 2.2, there are polynomials $P_m(X)$, $P_m^{(+)}(X)$ with integral coefficients such that

$$\zeta(s,K_m)=\frac{P_m(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$
$$\zeta(s,K_m^+)=\frac{P_m^{(+)}(q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

Put $P_m^{(-)}(X) = P_m(X)/P_m^{(+)}(X)$, then we have

$$\zeta^{(-)}(s, K_m) = P_m^{(-)}(q^{-s}). \tag{11}$$

Notice that the fields $K_m,\ K_m^+$ associate to $X_m,\ X_m^+$, respectively. Since any prime in K_m^+ above P_∞ is totally ramified in K_m/K_m^+, we have

$$P_m^{(-)}(q^{-s}) = \prod_{\chi \in X_m^-} L(s, \chi) \tag{12}$$

where $X_m^- = X_m - X_m^+$.

The L-function associated to the non-trivial character can be expressed by the polynomial of q^{-s} with complex coefficients. Hence we see that $P_m^{(-)}(X)$ is the polynomial with integral coefficients.

3 The determinant formula for $P_m^{(-)}(X)$

In the previous section, we defined the relative congruence zeta function $\zeta^{(-)}(s, K_m)$ for the m-th cyclotomic function field, and we showed that $\zeta(s, K_m)$ is expressed by the polynomial $P_m^{(-)}(X)$ with integral coefficients. The goal of this section is to give a determinant formula for $P_m^{(-)}(X)$. First, we will prepare some notations to construct the determinant formula.

Let m be a monic polynominal of degree d. For $\alpha \in (A/(m))^\times$, there is a unique element of $r_\alpha \in A$ satisfying

$$\begin{aligned} r_\alpha &= a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 \quad (n = \deg r_\alpha < d), \\ r_\alpha &\equiv \alpha \mod m, \end{aligned}$$

where $\deg f$ denotes the degree of the polynomial f. Then we define

$$\mathrm{Deg}(\alpha) = n, \quad L(\alpha) = a_n \in \mathbb{F}_q^\times$$

and $c^\lambda(\alpha) = \lambda^{-1}(L(\alpha))$ for the character λ of $\mathbb{F}_q^\times$. Put $N_m = \Phi(m)/(q-1)$. Let $\alpha_1, \alpha_2, \cdots \alpha_{N_m}$ be all of the elements of $(A/(m))^\times$ with $L(\alpha) = 1$, which are the complete system of representatives for $\mathcal{R}_m = (A/(m))^\times/\mathbb{F}_q^\times$. We put

$$\begin{aligned} c_{ij}^\lambda &= c^\lambda(\alpha_i \alpha_j^{-1}) \ \ (i, j = 1, 2, ..., N_m), \\ d_{ij} &= \mathrm{Deg}(\alpha_i \alpha_j^{-1}) \ \ (i, j = 1, 2, ..., N_m). \end{aligned}$$

For any character λ of $\mathbb{F}_q^\times$, we define the matrix

$$D_m^{(\lambda)}(X) = (c_{ij}^\lambda X^{d_{ij}})_{i,j=1,2,\ldots,N_m}.$$

The following matrix plays an essential role in our argument

$$D_m^{(-)}(X) = \prod_{\lambda \neq 1} D_m^{(\lambda)}(X) \tag{13}$$

where the product runs over all non-trivial characters of $\mathbb{F}_q^\times$. Notice that $d_{ij} > 0$ in the case $i \neq j$, and $d_{ij} = 0,\ c_{ij}^\lambda = 1$ in the case $i = j$. Thus $D_m^{(-)}(0)$ is the unit matrix. To state the main result, we prepare the polynomial $J_m^{(-)}(X)$ defined by

$$J_m^{(-)}(X) = \prod_{\chi \in X_m^-} \prod_{Q|m} (1 - \chi(Q) X^{\deg Q}) \tag{14}$$

where Q is an irreducible monic polynomial dividing m. To begin with, we prove the following proposition.

Proposition 3.1. *In the above notations, we have*

$$J_m^{(-)}(X) = \prod_{Q|m} \frac{(1 - X^{f_Q \deg Q})^{g_Q}}{(1 - X^{f_Q^+ \deg Q})^{g_Q^+}} \tag{15}$$

where f_Q, f_Q^+ are the residue class degrees of Q in $K_m/k, K_m^+/k$ respectively, and g_Q, g_Q^+ are the numbers of primes in K_m, K_m^+ respectively over Q.

Proof. Notice that $X_m,\ X_m^+$ associate to the m-th cyclotomic function field K_m, and its maximal real subfield K_m^+ respectively. Let Q be an irreducible monic polynomial dividing m. Put

$$Y_Q = \{\ \chi \in X_m \mid \chi(Q) \neq 0\ \},\ \ Z_Q = \{\ \chi \in X_m \mid \chi(Q) = 1\ \}.$$

From Theorem 2.1, we have

$$\begin{aligned}
\prod_{\chi \in X_m} (1 - \chi(Q) X^{\deg Q}) &= \prod_{\chi \in Y_Q} \left(1 - \chi(Q) X^{\deg Q}\right) \\
&= \prod_{\chi \in Y_Q/Z_Q} \prod_{\psi \in Z_Q} \left(1 - \chi\psi(Q) X^{\deg Q}\right) \\
&= \Big(\prod_{\chi \in Y_Q/Z_Q} \left(1 - \chi(Q) X^{\deg Q}\right) \Big)^{g_Q}.
\end{aligned}$$

Since Y_Q/Z_Q is a cyclic group of order f_Q, we have

$$\prod_{\chi \in Y_Q/Z_Q} \left(1 - \chi(Q) X^{\deg Q}\right) = \left(1 - X^{f_Q \deg Q}\right).$$

Hence we obtain

$$\prod_{\chi \in X_m} \left(1 - \chi(Q) X^{\deg Q}\right) = \left(1 - X^{f_Q \deg Q}\right)^{g_Q}. \tag{16}$$

By the same argument, we have

$$\prod_{\chi \in X_m^+} \left(1 - \chi(Q) X^{\deg Q}\right) = \left(1 - X^{f_Q^+ \deg Q}\right)^{g_Q^+}. \tag{17}$$

Noting that $X_m^- = X_m - X_m^+$, we can get the proposition from the above equations (16), (17). $\square$

There are several consequences of this proposition. First of all, by Proposition 3.1, we see that $J_m^{(-)}(X)$ is a polynomial with integral coefficients. Secondly, if m is the power of an irreducible polynomial P, the prime P is totally ramified in K_m/k (cf. [Ro 2]). Hence we obtain $J_m^{(-)}(X) = 1$ in this case.

The next theorem is our main result of the present paper.

Theorem 3.1. *Let $m \in A$ be a monic polynomial. Then, we have*

$$\det D_m^{(-)}(X) = P_m^{(-)}(X) J_m^{(-)}(X). \tag{18}$$

Proof. For any $\chi \in X_m$, let the monic polynomial f_χ be the conductor of χ. Define $\tilde{\chi}$ by

$$\tilde{\chi} = \chi \circ \pi_\chi$$

where $\pi_\chi : (A/(m))^\times \to (A/(f_\chi))^\times$ is the natural homomorphism. Then, we have

$$L(s, \tilde{\chi}) = L(s, \chi) \cdot \prod_{Q|m} \left(1 - \chi(Q) q^{-s \deg Q}\right). \tag{19}$$

Fix a non-trivial character λ of $\mathbb{F}_q^\times$, and $\psi \in X_m^-$ $(\psi|_{\mathbb{F}_q^\times} = \lambda)$. Then we have

$$\psi \cdot X_m^+ = \{\chi \in X_m^- \mid \chi|_{\mathbb{F}_q^\times} = \lambda\}.$$

For a character $\chi \in X_m^-$ ($\chi|_{\mathbb{F}_q^\times} = \lambda$), there is a unique character $\phi \in X_m^+$ with $\chi = \psi \cdot \phi$. By the same argument as in Lemma 3 in [G-R],

$$\begin{aligned} L(s, \tilde{\chi}) &= \sum_{i=1}^{N_m} \tilde{\chi}(\alpha_i) q^{-\mathrm{Deg}(\alpha_i)s} \\ &= \sum_{i=1}^{N_m} \tilde{\phi}(\alpha_i) \tilde{\psi}(\alpha_i) c^\lambda(\alpha_i) q^{-\mathrm{Deg}(\alpha_i)s}. \end{aligned}$$

Notice that $\tilde{\psi}(\alpha) c^\lambda(\alpha)$ and Deg are functions over $\mathcal{R}_m$, and $\tilde{\phi}$ runs through all characters of $\mathcal{R}_m$ when ϕ runs through all characters of X_m^+. By the Frobenius determinant formula (cf. [Wa], Lemma 5.26),

$$\begin{aligned} \prod_{\chi|_{\mathbb{F}_q^\times} = \lambda} L(s, \tilde{\chi}) &= \prod_{\phi \in X_m^+} \sum_{i=1}^{N_m} \tilde{\phi}(\alpha_i) \tilde{\psi}(\alpha_i) c^\lambda(\alpha_i) q^{-\mathrm{Deg}(\alpha_i)s} \\ &= \det(\psi(\alpha_i \alpha_j^{-1}) c_{ij}^\lambda q^{-s d_{ij}})_{i,j=1,2,\cdots N_m} \\ &= \det D_m^{(\lambda)}(q^{-s}). \end{aligned}$$

From the decomposition

$$X_m^- = \bigcup_{\lambda \neq 1} \{\chi \in X_m \mid \chi|_{\mathbb{F}_q^\times} = \lambda\},$$

we have

$$\det D_m^{(-)}(q^{-s}) = \prod_{\chi \in X_m^-} L(s, \chi) \cdot J_m^{(-)}(q^{-s}).$$

By equation (12), we obtain

$$\det D_m^{(-)}(q^{-s}) = P_m^{(-)}(q^{-s}) J_m^{(-)}(q^{-s}). \tag{20}$$

Putting $X = q^{-s}$, we obtain the desired result. □

We give two remarks of this theorem. To begin with, $P_m^{(-)}(X) = 1$ when m is the monic polynomial of degree 1. In fact, we calculate $D_m^{(-)}(X) = 1$ in this case. Secondly, recall $J_m^{(-)}(X) = 1$ when m is the power of an irreducible polynomial. Hence $D_m^{(-)}(X) = P_m^{(-)}(X)$ in this case.

As a special case of our result, we obtain the following determinant formula for relative class numbers.

Corollary 3.1. (cf. [B-K], [A-C-J]) *Let h_m^- be the relative class number of K_m. Put $f_Q^- = f_Q/f_Q^+$ and $g_Q^- = g_Q/g_Q^+$, then*

$$\prod_{\lambda \neq 1} \det(c_{ij}^{\lambda})_{i,j=1,2,\ldots,N_m} = W_m^- \cdot h_m^- \tag{21}$$

where

$$W_m^- = \begin{cases} \prod_{Q|m} (f_Q^-)^{g_Q^+} & \textit{if } g_Q^- = 1 \textit{ for every prime } Q \textit{ dividing } m, \\ 0 & \textit{otherwise.} \end{cases} \tag{22}$$

Proof. Putting $X = 1$ in Theorem 3.1, we see that

$$\det D_m^{(-)}(1) = \prod_{\lambda \neq 1} \det(c_{ij}^{\lambda}), \tag{23}$$

and $J_m^{(-)}(1) = W_m^-$ by Proposition 3.1. Since $P_m^{(-)}(1) = h_m^-$, we obtain the desired result. □

If m is the power of an irreducible polynomial, we see that $W_m^- = 1$. In other case, any prime in K_m^+ except infinite primes is not ramified in K_m/K_m^+. Thus we see $f_Q^- = q - 1$ for the prime Q with $g_Q^- = 1$.

4 Some coefficients of low degree terms of $\det D_m^{(-)}(X)$

In this section, we will calculate the coefficients of $\det D_m^{(-)}(X)$ of degree 1, 2 by using the derivative of determinant.

Let $m \in A$ be a monic polynomial. Noting $\det D_m^{(-)}(0) = 1$, we see that $\det D_m^{(-)}(X)$ can be written by

$$\det D_m^{(-)}(X) = 1 + a_1 X + a_2 X^2 + \cdots \tag{24}$$

where a_i $(i = 1, 2, ...)$ are integers.

Proposition 4.1. *Let $m \in A$ be a monic polynomial of degree d (> 1). Then, we have*

$$(1)\ a_1 = 0, \tag{25}$$

$$(2)\ a_2 = 0 \quad (\textit{if } \deg m > 2), \tag{26}$$

$$(3)\ a_2 = \frac{N_m}{2}\{(q-1)(1 - C_m) + N_m - 1\} \quad (\textit{if } \deg m = 2), \tag{27}$$

where

$$C_m = {}^{\#}\{i = 1, 2, ..., N_m \mid L(\alpha_i^{-1}) = 1\}. \tag{28}$$

Here ${}^{\#}A$ *is the number of elements of a set* A.

By Proposition 3.1, we can obtain $J_m^{(-)}(X)$. Hence we can also calculate coefficients of low degree terms of $P_m^{(-)}(X)$.
To prove Proposition 3.1, we first state the next lemma, which can be shown by simple calculations.

Lemma 4.1. *Let* $F(X) = (f_{ij}(X))_{i,j}$ *be a matrix with one variable. If* $F(X)$ *is twice differentiable and invertible at* $X = X_0$*, then*

$$\begin{aligned}
(1)\ \left.\frac{d\ \det F(X)}{dX}\right|_{X=X_0} &= \det F(X_0) \cdot \mathrm{Tr}\Big(F(X_0)^{-1}\frac{dF}{dX}(X_0)\Big),\\
(2)\ \left.\frac{d^2 \det F(X)}{dX^2}\right|_{X=X_0} &= \det F(X_0) \cdot \Big\{ \mathrm{Tr}\Big(F(X_0)^{-1}\frac{d^2F}{dX^2}(X_0)\Big) - \\
&\quad \mathrm{Tr}\Big(F(X_0)^{-1}\frac{dF}{dX}(X_0)F(X_0)^{-1}\frac{dF}{dX}(X_0)\Big) + \\
&\quad \mathrm{Tr}\Big(F(X_0)^{-1}\frac{dF}{dX}(X_0)\Big)^2 \Big\},
\end{aligned}$$

where $\mathrm{Tr}(A)$ *is the trace of the matrix* A.

Now we prove the proposition.

Proof. Let λ be a non-trivial character of $\mathbb{F}_q^\times$, and write

$$\det D_m^{(\lambda)}(X) = 1 + a_1^\lambda X + a_2^\lambda X^2 + \cdots .$$

Notice that $D_m^{(\lambda)}(0)$ is the unit matrix and

$$\frac{dD_m^{(\lambda)}}{dX}(0) = (l_{ij})_{i,j=1,2,\cdots,N_m} \tag{29}$$

where

$$l_{ij} = \begin{cases} 0 & \text{if } d_{ij} = 0 \text{ or } d_{ij} > 1, \\ c_{ij}^\lambda & \text{if } d_{ij} = 1\ . \end{cases} \tag{30}$$

By Lemma 4.1, $a_1^\lambda = 0$ and

$$a_2^\lambda = -\frac{1}{2}\mathrm{Tr}\Big(\Big(\frac{dD_m^{(\lambda)}}{dX}(0)\Big)^2\Big).$$

Thus we have assertion (1). If $\deg m > 2$, there is no combination (i, j) such that $d_{ij} = 1$, and $d_{ji} = 1$. Thus we have $a_2^\lambda = 0$ in the case $\deg m > 2$. Since $a_2 = \sum_{\lambda \neq 1} a_2^\lambda$, we obtain assertions (2).

Next we prove the case when $\deg m = 2$. In this case, we have

$$l_{ij} = \begin{cases} 0 & \text{if } i = j, \\ c_{ij}^\lambda & \text{if } i \neq j. \end{cases} \tag{31}$$

Thus we have

$$\begin{aligned}\sum_{\lambda \neq 1} a_2^\lambda &= \sum_{\lambda \neq 1}\Big(\frac{N_m}{2} - \frac{1}{2}\sum_{i=1}^{N_m}\sum_{j=1}^{N_m} \lambda^{-1}(L(\alpha_i\alpha_j^{-1})L(\alpha_j\alpha_i^{-1}))\Big) \\ &= \frac{N_m(q-2)}{2} - \frac{1}{2}\sum_{i=1}^{N_m}\sum_{j=1}^{N_m} e_{ij}\end{aligned}$$

where

$$e_{ij} = \begin{cases} q-2 & \text{if } L(\alpha_i\alpha_j^{-1})L(\alpha_j\alpha_i^{-1}) = 1, \\ -1 & \text{otherwise.} \end{cases}$$

For any $i, j \in \{1, 2, \cdots N_m\}$, there are $\gamma_{ij} \in \mathbb{F}_q^\times$ and $\beta_{ij} \in (A/(m))^\times$ with $L(\beta_{ij}) = 1$ such that $\alpha_i\alpha_j^{-1} = \gamma_{ij}\beta_{ij}$. Then we have

$$L(\alpha_i\alpha_j^{-1})L(\alpha_j\alpha_i^{-1}) = L(\beta_{ij}^{-1}).$$

Noting

$$\{\beta_{ij} \mid j = 1, 2, \cdots N_m\} = \{\alpha_j \mid j = 1, 2, \cdots N_m\},$$

we have

$$\sum_{j=1}^{N_m} e_{ij} = (q-1)C_m - N_m.$$

Thus we have the desired result. □

We consider the case when $m = T^2 + aT + b \in A$. If $\alpha = T - c$ satisfies $L(\alpha^{-1}) = 1$, then c is a root of the equation $T^2 + aT + b + 1$. Thus we obtain $C_m \leqq 3$.

5 Examples

In this section, we give some examples.

Example 5.1. *For $q = 3$ and $m = T^2 + 1$, we see that the extension degree of K_m/k is 8, and $N_m = 4$. Since the polynomial m is irreducible, we have* $\det D_m^{(-)}(X) = P_m^{(-)}(X)$. *Put*

$$\alpha_1 = 1,\ \alpha_2 = T,\ \alpha_3 = T + 1,\ \alpha_4 = T + 2.$$

Then we have

$$\begin{aligned} P_m^{(-)}(X) &= \det D_m^{-}(X) \\ &= \begin{vmatrix} 1 & -X & X & X \\ X & 1 & -X & X \\ X & -X & 1 & -X \\ X & X & X & 1 \end{vmatrix} \\ &= 1 - 2X^2 + 9X^4. \end{aligned}$$

The relative class number h_m^- of K_m is $P_m^{(-)}(1) = 8$.

Example 5.2. *For $q = 3$ and $m = T^3 + T^2$, we see that the extension degree of K_m/k is 12, and $N_m = 6$. Put*

$$\begin{aligned} \alpha_1 &= 1,\ \alpha_2 = T^2 + 2T + 2,\ \alpha_3 = T^2 + T + 1, \\ \alpha_4 &= T + 2,\ \alpha_5 = T^2 + 1\ \ \alpha_6 = T^2 + T + 2. \end{aligned}$$

Then we have

$$\begin{aligned} \det D_m^{(-)}(X) &= \begin{vmatrix} 1 & X & -X^2 & X^2 & X^2 & -X^2 \\ X^2 & 1 & -X^2 & -X^2 & -X^2 & -X \\ X^2 & X^2 & 1 & X & -X^2 & X^2 \\ X & X^2 & X^2 & 1 & X^2 & X^2 \\ X^2 & X^2 & -X & -X^2 & 1 & X^2 \\ X^2 & -X^2 & -X^2 & X^2 & X & 1 \end{vmatrix} \\ &= 1 - 6X^3 - 3X^4 - 6X^5 + 23X^6 + 30X^7 + 6X^8 - 18X^9 - 27X^{10}, \end{aligned}$$

and

$$J_m^{(-)}(X) = 1 + X - X^3 - X^4.$$

Thus we obtain

$$\begin{aligned} P_m^{(-)}(X) &= \frac{\det D_m^{(-)}(X)}{J_m^{(-)}(X)} \\ &= 1 - X + X^2 - 6X^3 + 3X^4 - 9X^5 + 27X^6. \end{aligned}$$

The relative class number h_m^- *of* K_m *is* $P_m^{(-)}(1) = 16$.

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